

# Instantons in curvilinear coordinates

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## Abstract

The multi-instanton solutions by 'tHooft and Jackiw, Nohl & Rebhi are generalized to curvilinear coordinates. Expressions can be notably simplified by the appropriate gauge transformation. This generates the compensating addition to the gauge potential of pseudoparticles. Singularities of the compensating connection are irrelevant for physics but affect gauge dependent quantities.

## The third connection

The years that passed since the discovery of instantons, [1], did not bring answer to the question about the role of instantons in QCD, [2, 3]. As far as confinement remains a puzzle all references to instantons at long scales are ambiguous. Indications may come from studies of instanton effects in phenomenological models. These could tell whether confinement may seriously affect pseudoparticles and *v. v.*

Common confinement models look most natural in non-Cartesian coordinate frames. The obvious choice for bags are 3+1-cylindrical, *i. e.* 3-spherical+time, coordinates while strings would prefer 2+2-cylindrical (2+1-cylindrical+time) geometry. Nevertheless instantons were usually discussed in the Cartesian frame (that was ideal in vacuum). The purpose of the present work is to draw attention to the problem and to develop the adequate technique. We shall generalize to curvilinear coordinates the multi-instanton solutions by 'tHooft and Jackiw, Nohl & Rebhi, [4], and simplify the formulae by the gauge transformation. Presently I don't know whether the procedure is good for other topological configurations<sup>†</sup> but I would expect that it makes sense for the AHDM<sup>‡</sup>, [5], solution<sup>1</sup>.

We start from the basics of curvilinear coordinates in Sect. 1.1 and introduce the first two connections, namely the Levi-Civita connection and the spin connection. In Sect. 1.2 we describe the multi-instanton solutions. In Sect. 2 we shall rewrite instantons in non-Cartesian coordinates and propose the gauge transform that makes formulae compact. The price will be the appearance of the *third, so called compensating*, gauge connection. The example of the  $O(4)$ -spherical coordinates is sketched in Sect. 3. Singularities of the gauged solution are discussed in Sect. 4. The last part summarizes the results.

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\*The work is done with partial support of the RFBR grant 97-02-16131.

<sup>1</sup>I'm grateful to L. Lipatov<sup>†</sup> and S. Moch<sup>‡</sup> for the questions.

# 1 Basics

## 1.1 Curvilinear coordinates

We shall consider flat 4-dimensional euclidean space-time that may be parametrized either by the set of Cartesian coordinates  $x^\mu$  or by curvilinear ones called  $q^\alpha$ . The  $q$ -frame is characterized by the metric tensor  $g_{\alpha\beta}(q)$ :

$$ds^2 = dx_\mu^2 = g_{\alpha\beta}(q) dq^\alpha dq^\beta. \quad (1)$$

In the  $q$ -frame the derivatives  $\frac{\partial}{\partial x^\mu}$  should be replaced by the covariant ones,  $D_\alpha$ . For example the derivative of a covariant vector  $A_\beta$  is:

$$D_\alpha A_\beta = \partial_\alpha A_\beta - \Gamma_{\alpha\beta}^\gamma A_\gamma. \quad (2)$$

The function  $\Gamma_{\beta\gamma}^\alpha$  is called the **Levi-Civita connection**. It can be expressed in terms of the metric tensor ( $g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma$ ):

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha\delta} \left( \frac{\partial g_{\delta\beta}}{\partial q^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial q^\beta} - \frac{\partial g_{\beta\gamma}}{\partial q^\delta} \right). \quad (3)$$

Often it is convenient to use instead of  $g_{\alpha\beta}$  the four vectors  $e_\alpha^a$  called the vierbein:

$$g_{\alpha\beta}(q) = \delta_{ab} e_\alpha^a(q) e_\beta^b(q). \quad (4)$$

Multiplication by  $e_\alpha^a$  converts coordinate (Greek) indices into the vierbein (Latin) ones,

$$A^a = e_\alpha^a A^\alpha. \quad (5)$$

Covariant derivatives of quantities with Latin indices are defined in terms of the **spin connection**  $R_{\alpha b}^a(q)$ ,

$$D_\alpha A^a = \partial_\alpha A^a + R_{\alpha b}^a A^b. \quad (6)$$

The two connections  $\Gamma_{\alpha\delta}^\beta$  and  $R_{\alpha b}^a$  are related to each other as follows:

$$R_{\alpha b}^a = e_\beta^a \partial_\alpha e_b^\beta + e_\beta^a \Gamma_{\alpha\gamma}^\beta e_b^\gamma = e_\beta^a (D_\alpha e^\beta)_b. \quad (7)$$

## 1.2 Instantons

We shall discuss pure euclidean Yang-Mills theory with the  $SU(2)$  gauge group. The vector potential is  $\hat{A}_\mu = \frac{1}{2} \tau^a A_\mu^a$  where  $\tau^a$  are the Pauli matrices. The (Cartesian) covariant derivative is  $D_\mu = \partial_\mu - i \hat{A}_\mu$ , and the action has the form:

$$S = \int \frac{\text{tr } \hat{F}_{\mu\nu}^2}{2g^2} d^4x = \int \frac{\text{tr } \hat{F}_{\alpha\beta} \hat{F}^{\alpha\beta}}{2g^2} \sqrt{g} d^4q. \quad (8)$$

where  $g = \det ||g_{\alpha\beta}||$ . The formula for the gauge field strength  $\hat{F}_{\alpha\beta}$  is universal:

$$\hat{F}_{\alpha\beta}(\hat{A}) = \partial_\alpha \hat{A}_\beta - \partial_\beta \hat{A}_\alpha - i [\hat{A}_\alpha, \hat{A}_\beta]. \quad (9)$$

The action is invariant under gauge transformations,

$$\hat{A}_\mu \rightarrow \hat{A}_\mu^\Omega = \Omega^\dagger \hat{A}_\mu(x) \Omega + i \Omega^\dagger \partial_\mu \Omega, \quad (10)$$

where  $\Omega$  is a unitary  $2 \times 2$  matrix,  $\Omega^\dagger = \Omega^{-1}$ .

The field equations have selfdual ( $F_{\mu\nu} = \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} F^{\lambda\sigma}$ ) solutions known as instantons. The most general explicit selfdual configuration found by Jackiw, Nohl and Rebbi, [4], is:

$$\hat{A}_\mu(x) = -\frac{\hat{\eta}_{\mu\nu}^-}{2} \partial_\nu \ln \rho(x), \quad (11)$$

where  $\hat{\eta}_{\mu\nu}$  is the matrix version of the 'tHooft's  $\eta$ -symbol, [6]:

$$\hat{\eta}_{\mu\nu}^\pm = -\hat{\eta}_{\nu\mu}^\pm = \begin{cases} \tau^a \epsilon^{a\mu\nu}; & \mu, \nu = 1, 2, 3; \\ \pm \tau^a \delta^{\mu a}; & \nu = 4. \end{cases} \quad (12)$$

The widely used regular and singular instanton gauges as well as the famous 'tHooft's *Ansatz* may be cast into the form similar to (11).

Our aim is to generalize the solution (11) to curvilinear coordinates. We shall not refer to the explicit form of  $\rho(x)$  and the results will be applicable to all the cases.

## 2 Multi-instantons in curvilinear coordinates

### 2.1 The problem and the solution

It is not a big deal to transform the covariant vector  $\hat{A}_\mu$ , (11), to  $q$ -coordinates. However this makes the constant numerical tensor  $\hat{\eta}_{\mu\nu}$  coordinate-dependent

$$\hat{\eta}_{\mu\nu} \rightarrow \hat{\eta}_{\alpha\beta} = \hat{\eta}_{\mu\nu} \frac{\partial x^\mu}{\partial q^\alpha} \frac{\partial x^\nu}{\partial q^\beta}. \quad (13)$$

We propose to factorize the coordinate dependence by means of the gauge transformation such that

$$\Omega^\dagger \hat{\eta}_{\alpha\beta} \Omega = e_\alpha^a e_\beta^b \hat{\xi}_{ab}. \quad (14)$$

Here  $\hat{\xi}_{ab}$  is a constant numerical matrix tensor,

$$\hat{\xi}_{ab} = \delta_a^\mu \delta_b^\nu \hat{\eta}_{\mu\nu}. \quad (15)$$

It takes the place of  $\hat{\eta}_{\mu\nu}$  in non-Cartesian coordinates.

It can be shown that the matrix  $\Omega$  does exist provided that the  $x$ -frame and the  $q$ -frame have the same orientation and the two sides of (14) are of same duality.

The gauge-rotated instanton field is the sum of the two pieces:

$$\hat{A}_\alpha^\Omega(q) = -\frac{1}{2} e_\alpha^a \hat{\xi}_{ab} e^{b\beta} \partial_\beta \ln \rho(q) + i \Omega^\dagger \partial_\alpha \Omega. \quad (16)$$

The first addend is almost traditional and does not depend on the  $\Omega$ -matrix whereas the second one carries information about the  $q$ -frame. It is entirely of geometrical origin. We call it the **compensating connection** because it compensates the coordinate dependence of  $\hat{\eta}_{ab} = e_a^\alpha e_b^\beta \hat{\eta}_{\alpha\beta}$  and reduces it to the constant  $\hat{\xi}_{ab}$ .

So long we did not specify what was the duality of the  $\hat{\eta}$ -symbol. However the  $\Omega$ -matrices and compensating connections for  $\hat{\eta}^+$  and  $\hat{\eta}^-$  are different. In general  $\hat{A}_\alpha^{\text{comp}\pm}$  are respectively the selfdual and antiselfdual projections of the spin connection onto the gauge group:

$$\hat{A}_\alpha^{\text{comp}\pm} = i \Omega_\pm^\dagger \partial_\alpha \Omega_\pm = -\frac{1}{4} R_\alpha^{ab} \hat{\xi}_{ab}^\pm. \quad (17)$$

The last formula does not contain  $\Omega$  that has dropped out of the final result. In order to write down the multi-instanton solution one needs only the vierbein and the associated spin connection.

## 2.2 Triviality of the compensating field.

The fact of the compensating connection  $\hat{A}^{\text{comp}}$ , (17), being a pure gauge is specific to the flat space. It turns out that the field strength  $\hat{F}_{\alpha\beta}(\hat{A}^{\text{comp}})$  is related to the Riemann curvature of the space-time  $R_{\alpha\beta}^{\gamma\delta}$ :

$$\hat{F}_{\alpha\beta}(\hat{A}^{\text{comp}\pm}) = -\frac{1}{4} R_{\alpha\beta}^{\gamma\delta} \hat{\xi}_{\gamma\delta}^\pm. \quad (18)$$

Thus  $\hat{F}_{\alpha\beta}(\hat{A}^{\text{comp}}) = 0$  provided that  $R_{\alpha\beta}^{\gamma\delta} = 0$ . Simple changes of variables  $x^\mu \rightarrow q^\alpha$  do not generate curvature and  $\hat{A}^{\text{comp}}$  is a pure gauge. However this is not the case in curved space-times.

## 2.3 Duality and topological charge

As long as we limit ourselves to identical transformations the vector potential (16) must satisfy the classical field equations. However the duality equation looks differently in non-Cartesian frame. If written with coordinate indices it is:

$$\hat{F}_{\alpha\beta} = \frac{\sqrt{g}}{2} \epsilon_{\alpha\beta\gamma\delta} \hat{F}^{\gamma\delta}. \quad (19a)$$

Still it retains the familiar form in the vierbein notation:

$$\hat{F}_{ab} = \frac{1}{2} \epsilon_{abcd} \hat{F}^{cd}. \quad (19b)$$

The topological charge is given by the integral

$$q = \frac{1}{32\pi^2} \int \epsilon_{\alpha\beta\gamma\delta} \text{tr} \hat{F}^{\alpha\beta} \hat{F}^{\gamma\delta} d^4q, \quad (20a)$$

which in the vierbein notation becomes

$$q = \frac{1}{32\pi^2} \int \epsilon_{abcd} \text{tr} \hat{F}^{ab} \hat{F}^{cd} \sqrt{g} d^4q. \quad (20b)$$

The general expression for  $\hat{F}_{\alpha\beta}$  in non-Cartesian frame is rather clumsy but it simplifies for one instanton. The vector potential in regular gauge is, ( $r^2 = x_\mu^2$ ):

$$\hat{A}_\mu^I = \frac{\hat{\eta}_{\mu\nu}^+}{2} \partial_\nu \ln(r^2 + \rho^2). \quad (21)$$

The conjugated coordinate and  $\Omega_+$  gauge transformations convert it into

$$\hat{A}_\alpha^I = \frac{1}{2} e_\alpha^a \hat{\xi}_{ab} e^{b\beta} \partial_\beta \ln(r^2 + \rho^2) + \hat{A}^{\text{comp}+}, \quad (22)$$

and the field strength becomes plainly selfdual:

$$\hat{F}_{ab}(\hat{A}^I) = -\frac{2\hat{\xi}_{ab}^+}{(r^2 + \rho^2)^2}, \quad (23)$$

This generalizes regular gauge to any non-Cartesian coordinates.

### 3 Example

We shall consider an instanton placed at the origin of the  $O(4)$ -spherical coordinates. Those are the radius and three angles:  $q^\alpha = (\chi, \phi, \theta, r)$ . The polar axis is aligned with  $x^1$  and

$$x^1 = r \cos \chi; \quad (24a)$$

$$x^2 = r \sin \chi \sin \theta \cos \phi; \quad (24b)$$

$$x^3 = r \sin \chi \sin \theta \sin \phi; \quad (24c)$$

$$x^4 = r \sin \chi \cos \theta. \quad (24d)$$

The vierbein and the metric tensor are diagonal:

$$e_\alpha^a = \text{diag}(r, r \sin \chi \sin \theta, r \sin \chi, 1). \quad (25)$$

Now one may start from the vector potential (21) and consecutively carry out the entire procedure. But to the calculation of the instanton part this includes calculating  $\Gamma_{\beta\gamma}^\alpha$ , finding  $R_\alpha^{ab}$  and, finally, computing  $\hat{A}^{\text{comp}+}$ . The  $\hat{\xi}_{ab}^+$ -symbol coincides with  $\hat{\eta}_{ab}^+$ , (12, 15). The result is:

$$\hat{A}_\chi^I = \frac{\tau_x}{2} \left( \frac{r^2 - \rho^2}{r^2 + \rho^2} \right); \quad (26a)$$

$$\begin{aligned} \hat{A}_\phi^I &= -\frac{\tau_x}{2} \cos \theta + \frac{\tau_y}{2} \sin \chi \sin \theta \left( \frac{r^2 - \rho^2}{r^2 + \rho^2} \right) \\ &\quad + \frac{\tau_z}{2} \cos \chi \sin \theta; \end{aligned} \quad (26b)$$

$$\hat{A}_\theta^I = -\frac{\tau_y}{2} \cos \chi + \frac{\tau_z}{2} \sin \chi \left( \frac{r^2 - \rho^2}{r^2 + \rho^2} \right); \quad (26c)$$

$$\hat{A}_r^I = 0. \quad (26d)$$

The corresponding field strength is given by (23).

## 4 Singularities

Note that the vector field (26) is singular since neither  $\hat{A}_\theta^I$  nor  $\hat{A}_\phi^I$  goes to zero near polar axes  $\chi = 0$  and  $\theta = 0$ . These singularities are produced by the gauge transformation and must not affect observables. However they may tell on gauge variant quantities. We shall demonstrate that for the Chern-Simons number.

The topological charge, (20), can be represented by the surface integral,  $q = \oint K^\alpha dS_\alpha$ , [2, 3]. Here,

$$K^\alpha = \frac{\epsilon^{\alpha\beta\gamma\delta}}{16\pi^2} \text{tr} \left( \hat{A}_\beta \hat{F}_{\gamma\delta} + \frac{2i}{3} \hat{A}_\beta \hat{A}_\gamma \hat{A}_\delta \right). \quad (27)$$

Even though  $q$  is invariant  $K^\alpha$  depends on gauge. Consider Cartesian instanton in the  $\hat{A}_4 = 0$  gauge. The two contributions to the topological charge come from the  $x_4 = \pm\infty$  hyperplanes,  $q = N_{\text{CS}}(\infty) - N_{\text{CS}}(-\infty)$ , and the quantity

$$N_{\text{CS}}(t) = \int_{x_4=t} K^4 dS_4 \quad (28)$$

is called the Chern-Simon number. Instanton is a transition between two 3-dimensional vacua with  $\Delta N_{\text{CS}} = 1$ .

Analysis of (26) reveals a striking resemblance with this case. By coincidence here again  $\hat{A}_4 = 0$ , (26d). This gives an idea to interpret  $r$  as a time coordinate attributing the Chern-Simons number  $N_{\text{CS}}(r)$  to the sphere of radius  $r$ . A naive expectation would be that  $\Delta N_{\text{CS}} = N_{\text{CS}}(r)|_0^\infty$  gives the topological charge. However this is not true and  $\Delta N_{\text{CS}} = \frac{1}{2}$ . The second half of  $q$  is contributed by the singularities at  $\theta = 0, \pi$ . The  $\Omega$ -transform has affected the distribution of  $N_{\text{CS}}$ .

We conclude that in our approach gauge variant quantities depend on coordinate frame and may be localized at the singularities of the  $\Omega$ -transform. This may be one more way to simplify calculations with the help of curvilinear coordinates.

## Summary

We have shown that explicit (*multi*-)instanton solutions can be generalized to curvilinear coordinates. The gauge transformation converts the coordinate-dependent  $\hat{\eta}_{ab}$ -symbol into the constant  $\hat{\xi}_{ab}$ . The gauge potential is a sum of the instanton part and the compensating gauge connection, (16).

The compensating gauge connection can be computed in the three steps:

1. One starts from the calculation of the Levi-Civita connection  $\Gamma_{\beta\gamma}^\alpha$ , (3).
2. Covariant differentiation of the vierbein, (7), leads to the spin connection  $R_\alpha^{ab}$ .
3. Convolution of the spin connection with the appropriate  $\hat{\xi}_{ab}$  gives the compensating gauge potential, (17).

The advantage of our solution is that it is constructed directly of geometrical quantities, *i. e.* the vierbein and the spin connection. Another attractive feature is the relation between gauge variant quantities and the coordinate frame. More details may be found in [7].

# Acknowledgments

I would like to acknowledge financial support of the Ministry of Science of Russian Federation as well as that of the Organizing Committee which made possible my participation in the conference. It is a pleasure to thank the Chairman, professor S. Narison and the staff for personal care of participants.

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